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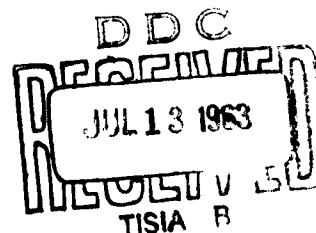
TN-479

METHODS FOR THE TIME SERIES ANALYSIS OF WATER WAVE  
EFFECTS ON PILES

14 June 1963

U. S. NAVAL CIVIL ENGINEERING LABORATORY  
Port Hueneme, California

409 966



## METHODS FOR THE TIME SERIES ANALYSIS OF WATER WAVE EFFECTS ON PILES

Lecture delivered by Dr. W. J. Pierson, Jr. at U. S. Naval Civil Engineering Laboratory, Port Hueneme, California on 8 June 1961.

Task No. Y-R011-01-051

Type C

by

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### ABSTRACT

The object of the task is to obtain coefficients of drag and mass for vertical circular piles in a hurricane wave environment by the analysis, using non-periodic techniques, of force and water level measurements versus time as obtained in the open Gulf of Mexico.

Three methods for the analysis of water wave effects on piles are outlined and compared, namely: bump-counting, time domain operations, and spectral operations. The computational requirements of the time-domain representation as introduced by Reid (1958) are contrasted with those required by the corresponding spectral representation. The joint distribution of the velocity ( $u$ ) and the acceleration ( $\dot{u}$ ) is given, from which the probability density function of the horizontal component of the force on the pile,  $f(t) = k_1 u u + k_2 \dot{u}$ , is derived where  $k_1$  and  $k_2$  are constants containing respectively coefficients of drag and mass. A detailed procedure for evaluating the probability density function is included.

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## INTRODUCTION

It is always very difficult to talk in this particular area to a group that you have not worked with before because technical work today has its own little secrets and magic words and my magic words may not be the magic words that you work with. If you try to avoid them in a presentation, it sometimes confuses your train of thought. So if at any time I use a word that you do not understand, the time to stop me is right then and there to see if I can give a simpler and more straightforward definition.\*

Techniques of time series analysis are applicable over a very wide range of problems in geophysics and engineering. They have been applied to the study of such things as the free oscillations of the earth; turbulence; noise in radio circuitry; and tides (so as to detect very weak truly periodic tides in a background of noise due to the meteorological variability). I have applied these techniques to problems in waves and ship motions. We have a group at New York University which includes Dr. Leo J. Tick that is doing some very interesting work in this area now. We are beginning to get to the point where we can do some intelligent thinking in connection with those problems that have non-linear features. The major structure of these theories today is growing dependent upon certain linearizing assumptions that can lead to incorrect results once in a while if conditions get to be extreme.

There are different levels of complexity with which a person can attack a particular problem. He can have in his mind when he starts, a fairly adequate idea of what he wants to get

\*The reader must of course, just look up the meaning of unfamiliar words.

out as the end product of his investigation or he can let the chips, if you will, fall where they may and hope that something drops out by luck at the end of the study. Usually, it is wise to be careful about deciding what you want before you start.

Suppose I want to study a ship moving through the waves. The ship is underway, under her own power, moving at a certain heading with a speed and direction that is constant except perhaps for such things as surge and a little oscillatory motion about the mean course. One could do a number of things to study the motion of the ship. Two airplanes -- helicopters probably -- could fly along exactly at certain positions over the ship and take stereo motion pictures of the sea surface all around the ship so that the exact wave elevation everywhere around the ship within say several hundred yards of the ship would be known. The ship could be instrumented to record heave, pitch, roll, surge, sway and yaw. If one wanted to get real fancy, one might even begin to worry about the effects of the gustiness of the wind on the superstructure. Miles and miles of chart paper and hundreds and hundreds of stereo pairs of photographs of the sea surface could be obtained. If you really wanted to do the work and if you had about 10 years to do it and a lot of data reduction facilities, and a great deal of patience, you could start with the stereo pattern of the sea surface and come up with a prediction of every one of the recorded motions of that vessel that would be very close to the observed motion. You would not have to mention the word probability; you would not have to mention the word statistics; and you would not have to mention the word spectra to do that job although spectral concepts would help in designing the computations to be made.

Now it is also suggested that a simpler problem could be done in about 1/100th of the time if you concede a certain point to me, namely, that you do not want the exact time histories that are recorded, you only want certain statistical features of it. All of these statistical features can be obtained from a proper analysis of the seaway in about 1/100th of the time with a corresponding saving in computational effort. But then one would have to use the concepts

of probability, statistics and spectra.

So this particular problem can be attacked two different ways. It is your decision as to what you want. Do you want faithful reproduction of everything that happens down to, if you will, a gnat's eyebrow, for this particular sequence of events, or do you want an adequately condensed summary of many of the pertinent features of this particular experiment?

There are other times when about the only thing you can do is try to predict the time history or explain a time history as it is observed, and then you have to go into the more detailed calculations that occur in the time domain. You cannot use these concepts of time series and spectra except as guides as to how to proceed.

Then there are other problems in which you do not even know how to do that. Then you have to resort to what I call "bump-counting." Bump-counting is a highly dangerous procedure and most of the time it has been used, to my knowledge, the end results are shown to have a large amount of scatter and considerable uncertainty and unreliability.

### THREE POSSIBLE TECHNIQUES

The three different techniques that are involved in trying to study such problems can be illustrated. These three techniques might be called (a) bump-counting, (b) time domain operations, and (c) spectral operations.

A problem that seems to be of interest here can be done by the three different ways and the results compared and contrasted. This particular problem has to do with wave forces on piles. We have a pile and it is supported as shown in Figure (1). The bottom and the free water surface are also shown. The problem will be simplified to the case where one feature of the sea surface will be that the waves are long crested. In other words, the variability in one space component will be neglected. This simplification could be serious in this problem. Just how serious is not known, but errors in

what is predicted for some of these quantities will occur. Everything else can be a very close facsimile to what occurs in nature.

In summary, Figure (1) shows this pile in the water. The waves are going by, and they are simply a function of  $x$  and  $t$ . If we record the waves right at the piling, we end up simply with a function of time,  $t$ , as the wave elevation right at the pile. Now, in this particular connection one is interested in the velocities of the fluid along the length of the pile at a number of points along the pile so let's focus our attention for this problem again at a particular depth that we can call,  $d$ .

We would like to know, for example,  $u(t)$ , the horizontal component of the water velocity, at that depth. We would also like to know  $\frac{\partial u(t)}{\partial t}$ , or  $\dot{u}(t)$ , and then finally we would like to know a function called

$$f(t) = k_1 |u| u + k_2 \dot{u} \quad (1)$$

Also, we might want something corresponding to  $P(t)$ , the pressure at that depth in the absence of the pile due to the waves.

Suppose that the water is 30 feet deep,  $h$  is 10 feet from the bottom and the waves have frequencies in them corresponding to a deep water wave length of 90 feet. Then  $u(t)$ ,  $\dot{u}(t)$ , and  $P(t)$ , will all be appreciable. The waves are considered to be periodic, and they are sufficiently low so that we can neglect such things as non-linear features of the wave profile. We all know how to get all these quantities if for example,

$$\eta(x,t) = A \cos(kx - \omega t) \quad (2)$$

where  $\eta(x,t)$  is the free surface at different times at the point of observation. (If you wish, put  $x = 0$ .) In equation (2),  $k$  is the wave number,  $2\pi/L$ , where  $L$  is the wave length. Since  $C = \left(\frac{gL}{2\pi} \tanh \frac{2\pi d}{L}\right)^{1/2}$  and since  $L = CT$ , it follows that

$$\omega = \frac{2\pi}{T} = \frac{C}{L} \frac{2\pi}{L} = Ck = \left(\frac{2\pi g}{L}\right)^{1/2} (\tanh kd)^{1/2} \text{ and that given}$$

$\omega$  and  $d$ ,  $k$  can be found. Tables of this function are given in various Beach Erosion Board publications. We can find  $k$  and also the  $u$  component of the velocity from classical wave theory,

$$u(t) = A\omega \frac{\cosh k(-d + D)}{\sinh kD} \cos(kx - \omega t) \quad (3)$$

$$\dot{u}(t) = A\omega^2 \frac{\cosh k(-d + D)}{\sinh kD} \sin(kx - \omega t) \quad (4)$$

$$P(t) = A \frac{\cosh k(-d + D)}{\sinh kD} \cos(kx - \omega t) \quad (5)$$

if  $A$  is in feet,  $u$  in feet/sec.,  $\dot{u}$  in feet/sec.<sup>2</sup>, and  $P$  is in feet of static water pressure. (We could put  $x = 0$  and still have everything correct at the point of observation.)

This is all very fine for a wave train with one discrete frequency in it, but then if we start looking at the data we get in a real situation; we get something that looks like Figure (2).

Bump counting. A bump-counter says, "Here is a wave this high and with this period. Let's substitute these two quantities into equations (3) and (4) to predict  $u$  and  $\dot{u}$ . Then substituting into equations (1), we will have the value,  $f(t)$ , for this particular peak of the time history." This is quite an assumption, is it not? I think you will agree that it's been done over and over in many different applications. For example, oceanographers did it to  $P(t)$  and wondered why everything didn't come out right; they were 20% off, or more in predicting  $\eta(t)$  from  $P(t)$ . There seemed to be some mystery as to just how deep the water was. A paper written many years ago concluded that the classical theory of hydrodynamics was not right. The theory of classical hydrodynamics was right; what was being done was wrong.

Time domain operations. The man who solved this problem correctly for wave forces on piles by means of computations in the time domain representation was Reid (1958) and the



group that works with him at Texas A&M. What did he do? Well, in very crude and elementary terms, he made a Fourier analysis of  $\eta(t)$ . Following this procedure leads to correct results as will be shown a little later.

A bit of information theory should come in here. What is the way to study such a continuous record properly with digital techniques? If one marks the record off in equally spaced time intervals in such a way that the sequence of points that you get when connected by a smooth curve reproduces the interesting detail of the record, then the points describe the record sufficiently well. In other words, one would not take a record and read it at intervals lettered I, II, III and so on in Figure (2). But at the same time one would not have to read one hundred points over the interval marked A to reproduce the important information in that record.

The record is inspected so as to estimate the highest frequency in the record (or the shortest period). The record is then read off at equally spaced time intervals equal to one half of this shortest period. The result is a sequence of numbers, and the sampling interval is said to have determined the "Nyquist frequency".

Let us call these numbers  $\eta_1, \eta_2, \eta_3$ , up to  $\eta_N$ . To reproduce a 20-minute ocean wave record of this nature adequately it is necessary to read it, depending on a lot of things, something like every second, perhaps every half second. Let us take a half second. There are then one hundred and twenty observations in a minute, or in a 20-minute record, about 2,400 numbers. Thus N in this case is about 2,400 numbers.

It is possible to write down a sum of sines and cosines that will reproduce this record, this whole 20-minute piece, exactly at all 2,400 points. There exists a theorem in information theory that says you can do it with a Fourier series that has 2,400 terms.

So, one would write down equation (6).

$$\eta(t) = \sum_{n=1}^{N/2} a_n \cos \frac{2\pi n t}{\tau} + \sum_{n=1}^{N/2} b_n \sin \frac{2\pi n t}{\tau} \quad (6)$$

Let the whole length of the record in Figure (2) be  $\tau$ , which is equal to 1200 seconds in this example. Then the Fourier series will contain all of the harmonics of  $\tau$ , namely,  $\tau/1, \tau/2, \tau/3, \dots, \tau/N$ . If we assume a zero mean so that the constant is zero, the periods in the record will be 1200 seconds, 600 seconds, 400 seconds, 300 seconds, 240 seconds, and so on all the way down to 1 second, if  $N$  is 2400. With this theory and with 1,199 values of  $a_n$  and 1,200 values of  $b_n$  (plus the zero mean to make a total of 2,400 values) for a 2,400 point record, one can reproduce this wiggly line exactly at the chosen points.

One might object that there is nothing in this record with a period of a second. That means that the  $a$ 's and  $b$ 's corresponding to this period are zero. It's just as important to know that some of the  $a$ 's and  $b$ 's are zero as to know that they have some value. The proper choice of spacing in Figure (2) for  $\eta_1, \eta_2, \dots, \eta_N$  would yield zero values for the high frequencies at the upper end of the frequency scale.

We have now reached the point where it is possible to reproduce the wave record at a chosen number of points exactly for any length of time. Outside of this interval of  $\tau$  seconds the actual record in nature will go on doing something similar that we do not care about. It will not look like Figure (2) anymore. The series representation in equation (6) will repeat itself for the next interval of  $\tau$  seconds. We do not care about that either if we consider only the record that was obtained.

Now let us predict  $u(t)$  from the  $\eta(t)$  function. What must we do? We must multiply each of these coefficients,  $a_n$  and  $b_n$  by  $\frac{\omega \cosh k(-d + D)}{\sinh kD}$  as in equation (3).

This process works just as well for the sine part as the

cosine part. But then notice that what happens to each coefficient is very strongly dependent on the frequency. So that when you study the  $u$  component of the velocity at a fairly great depth you do not see the high frequencies any more. One obtains a much smoother version of the free surface record. This is, of course, where one runs into trouble with the bump-counting technique because one cannot account for the selective attenuation of all the components in the record. Thus one can write down equations that correspond to  $u$ ,  $\dot{u}$ , and  $P(t)$  for this Fourier series. If one wanted to, these equations could be evaluated at each time point, and the records could be constructed for  $u(t)$ ,  $\dot{u}(t)$  and  $P(t)$  on the basis of this analysis. That isn't the way it would be done actually, but this is to show the mechanism behind it. The right attenuation, or the right factor, for each of these coefficients, is applied.

This attenuation depends on frequency, the depth of observation, and the depth of the water. (The wave number,  $k$ , is a function of frequency and water depth.) A graph

of the attenuation functions  $\frac{\omega \cosh k(-d + D)}{\sinh kD}$  and

$\frac{\omega^2 \cosh k(-d + D)}{\sinh kD}$  is shown in Figure (3), as the frequency ranges from zero to 2 radians per second for  $d = 74.5$  feet and  $D = 97.7$  feet.

A very different function, whose spectral components, these  $a$ 's and  $b$ 's for  $u$  and  $\dot{u}$ , are quite different from the coefficients of  $\eta$ , is the result when the Fourier series representations of  $u$  and  $\dot{u}$  are obtained. A large number of terms at the high frequency end where  $\eta$  has some contributions do not have to be evaluated when  $u$  and  $\dot{u}$  are considered. One might ask if the big (long in time) bumps are dominant. They are, but the big bumps have in them some of the very short waves too.

This is not the way that Reid did the problem; he went one step further and constructed the records of  $u$  and  $\dot{u}$

without ever finding the Fourier series representations. It is possible to apply the attenuation discussed above to each of the Fourier components and reconstruct the record. This procedure, described above, is easy to visualize, but it is not the easiest way to do it on a computer. However, it is equivalent to constructing  $u(t)$  by means of equation (7).

$$u(t) = \int_{-\infty}^{\infty} K(\tau^*) \eta(t - \tau^*) d\tau^* \quad (7)$$

The integral in (7) from  $-\infty$  to  $+\infty$  involves a kernel given by  $K(\tau^*)$ , and this kernel operates on  $\eta(t - \tau^*)$  to produce  $u(t)$ . ( $\tau^*$  has nothing to do with the  $\tau$  used before.) Equation (7) shows an integral from minus infinity to plus infinity. The actual integral extends over a range of perhaps plus or minus two minutes about the central point. In practice, the integration is replaced by a summation obtained by multiplying  $K(\tau^*)$  times  $(t - \tau^*)$  at a finite number of points and summing a group of such products. Each operation produces one point for  $u(t)$  and then  $t$  is increased to the next value needed and the process is repeated. There is another different kernel for  $\dot{u}$ .

The result is a problem for an electronic computer; one solves for what these kernels are, runs in  $\eta(t)$  and out comes  $u(t)$ . The kernel is a function, among other things, of the depth of the water and the depth of the point of observation. A new kernel is needed each time these values are changed. To do the same problem over again with the water twice as deep at a point still 15 feet below the surface, another kernel has to be calculated. To do it once again with the water twice as deep, the pile twice as long, and the point of observation twice as deep, still another kernel must be computed. The same thing can also be done for  $\dot{u}(t)$  except that another kernel is needed. The frequency effect is put in by the shape of the kernel.

Well, if you have seen some records of such observations, right away you can tell some of the things that happen. Some results of Lukasik and Groesch (1963) show such records. Other records show the next step where graphs of equation (1) are obtained. For study purposes, it might be very good to

try to measure  $u$  and  $\dot{u}$  at the pile before forming equation (1) to check how well each term is predicted.

Reid then compared the function,  $f(t)$ , that he formed mathematically by this prediction technique with the actual recorded observation of  $f(t)$ . The computed and predicted quantities agreed quite well. The agreement could be improved a little bit, but basic agreement was obtained between what was predicted numerically from the free surface record down to the point of observation, and the actual force recorded on the pile at that point.

Reid had a whole set of tables of these kernel functions and he applied them to the surface record and predicted the final output force. He didn't have any data to check  $u$  and  $\dot{u}$  against.

Now there was one trouble with Reid's study. In my opinion, it was that the records he used were not long enough. His results could probably have been improved if he had had records of 20 to 30 waves or more instead of 3 or 4 waves, although he has been able to predict total force. This is the only way he could predict these quantities for an observed wave record.

One may now ask the question: how does one generalize this and how would one tell an engineer what to do to use Reid's study to design a pile for some other place and for some other waves? The pile he used had strain gages so that one could determine the desired forces. Reid predicted the deflections and forces on it successfully. How would one apply this to an engineering problem? Now that one has reason to believe that the method works, one would take a free surface wave record at the point where a new installation is to be made and go through all these calculations on it. Except one wouldn't have experimental data to verify against anymore. A typical piece of time history of the forces on the pile would be enough in design considerations, would it not? It would be necessary to use experimental drag and mass coefficients to actually get the forces. So one could do it that way.

But now, supposing for some reason one wanted to know something about 200 different wave records and the effects of each of these wave records on a pile. By then one is beginning to get into quite a computational problem. Then you start counting pennies. The structure is worth so much, and the computation cost comes out to be 5 percent of the total cost. So we determine we can spend 5 percent of our money in designing and still save.

Spectral operations. But the time is going to come when one wants something a little simpler that will give essentially the same information, perhaps, and that would be where time series and statistics could come in.

As far as I know, this problem has not been solved in full in terms of the probability structure of the functions being studied. This is because of the non-linear term  $k|u|u$ . Now we haven't reached the point where we can do many things in non-linear theory from this point of view. However, let's just go back over some of this; for example, one would have no trouble at all in finding out everything one wants to know about  $\dot{u}$  or  $u$  on the basis of  $\eta(t)$  without predicting their exact time history. We might even be able to say a few things about equation (1) without predicting the exact time history.

The first thing one would do would be to take a look at the structure of equation (6). It has a very interesting structure.  $\eta(t)$ , as sketched, has a very interesting nature too and the  $a$ 's and  $b$ 's that describe  $\eta(t)$  have some very strange features.

First, consider all of the points that have been read off at equal time intervals through the whole record. If the waves are not too high, one can take all of these points and shuffle them so that their order is not known anymore and study them as a sample from some statistical population. These points turn out to be more or less normally distributed. The most probable value for points from this record is zero, or, stated another way, for any particular reading at some instant of time chosen at random, the value that is most probable is zero. More than that, the points from such a

sample have a normal distribution -- very closely, not exactly, but close enough for many practical applications.

A normal distribution is characterized by two numbers, the mean and the variance. If one is given these two numbers for a normal distribution, one knows everything about it. One can estimate the variance just from the points that have been read off. The record can be adjusted so that the points have a zero mean. All the values are squared, added, and divided by the total number of points. This gives the sample variance  $\hat{\psi}_0$  which can also be denoted as  $\hat{\sigma}^2$ . One feature of this record, namely,  $\hat{\psi}_0$ , as estimated is now known. (This corresponds to looking at

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [\eta(t)]^2 dt$$

and letting the time T get large in thinking about the actual true variance of the record.)

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{[\eta(t)]^2}{T} dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left( \sum_{n=1}^{N/2} a_n \cos \frac{2\pi n t}{T} + \sum_{n=1}^{N/2} b_n \sin \frac{2\pi n t}{T} \right)^2 dt \quad (8)$$

If one performs the same squaring operation on equation (6) as in (8), one sees right away that the sum of the squares of all the Fourier coefficients has to be similar to  $\psi_0$  also in some sense.

Now, one can talk about a function called a power spectrum. The power spectrum of  $\eta(t)$  is that function of the frequency,  $\omega$ , that resolves the total variance,  $\psi_0$ , into frequencies. Let us call the spectrum,  $S(\omega)$ . It has the property that

$$\int_b^\infty S(\omega) d\omega = \psi_0 \quad (9)$$

The total area under the spectral curve equals the variance of  $\eta(t)$ . Now imagine that one takes this record and puts a signal proportional to it on magnetic tape and imagine that one constructs a very narrow band filter, that passes only a certain amount of the total power in the signal. The filter is as shown in Figure (4).

Let's even be more idealistic about it and make a square filter. One cannot quite do it electronically, but let's assume that one has a very narrow band square filter. Let's put this electronic reproduction of the wave record through this filter and get out another record that is much purer in tone but which is not quite a sine wave. But all one can get are frequencies from  $\omega_0 - \frac{\Delta\omega}{2}$  to  $\omega_0 + \frac{\Delta\omega}{2}$ . Then

one computes the variance of this new record and plots it at  $\omega_0$ . Finally, let's tune this filter slowly over the whole range of  $\omega$ . The result is the power spectrum as a function of frequency as sketched in Figure 5. This spectrum is the resolution of the total variance of the wave record into its frequency components. This graph shows that most of the variance in the wave record is contained at  $\omega_A$ . There is nothing below  $\omega_B$ . The spectrum falls off asymptotically at high frequencies.

The term "power" spectrum is used because these concepts were first worked with by electrical engineers. They passed currents through resistances and  $I^2R$  is power, so this graph came out to be a power spectrum. But wave spectra and other spectra in civil engineering do not have the dimensions of power most of the time. For example, the spectrum of a wave record, which is a record of water elevation as a function of time, has the dimensions of  $(\text{length})^2 \times (\text{time})$ . Frequency is  $1/\text{time}$ ; therefore,  $S(\omega) d\omega$  has the dimensions of  $(\text{length})^2$  for the spectrum of the waves. The dimensions of the spectrum of  $u$  are  $[(\text{length})^2/(\text{time})^2] \times (\text{time})$ , or  $(L^2/T)$ . The dimensions of a "power" spectrum are given by the square of the ordinate, the vertical axis of whatever is being analyzed, times time. Thus spectra can have all kinds of oddball dimensions. It



depends upon the units of what is being measured. The dimension could be kips squared seconds for anchor chains, or feet per second, all squared, times seconds, or, for an acceleration, it could be feet per second squared, all squared, times seconds; i.e., ( $L^2T^{-3}$ ).

The "power" spectrum is to a stationary Gaussian process what the variance is to a normal distribution. That is, if one knows the variance of a normal distribution with a zero mean, one knows everything about the distribution. If one knows the spectrum of a stationary Gaussian process with a zero mean, "in principle", and one puts that in quotes, one knows everything about this stationary Gaussian process. Now there are lots of things that are not known yet, but, in principle, the spectrum is equivalent to the process for any problem in probability. Some of the problems of describing a stationary Gaussian process have not been solved, but in principle, given the spectrum the problem can be solved.

One can get a wide variety of useful information out of the spectrum without ever looking at the process. For example, one can predict the average time interval between zero up-crossings in a record. One can predict the number of maxima and minima in the record. One can even predict fairly well the distribution of the time intervals between zeros in the record. One can say something about the joint distribution of the amplitude of a half cycle and the half "period". All these results are very different from what your intuition would tell you and what the bump-counters obtain by these methods. They check very well in a number of applications. One can use the whole book by Rice (1944) on "The Mathematical Analysis of Random Noise" and apply it to these problems. Everything in it carries right over to the study of ocean wave records. It is a beautiful book written in 1944, full of very useful results that can be applied to ocean wave records.

So now one has the spectrum of the free surface, and one wishes to predict the spectrum of  $u$ . The spectrum of  $u$  is equal to the spectrum of the free surface times the square of the coefficient by which  $A$  in equation (2) is multiplied in (3), because that is the way each component

is attenuated to get  $u(t)$ . One is working with squared quantities and so if one takes the free surface spectrum and multiplies it by the square of this factor one obtains the spectrum of  $u$ . Let us see what the spectrum of  $u$  looks like. The factor can be written as

$$G(\omega) = \left[ \frac{\omega \cosh(k(\omega, d)[-d + D])}{\sinh k(\omega, d)D} \right]^2 \quad (10)$$

in order to emphasize the point that the wave number ( $k$ ) is a function of both frequency and depth.  $k = 2\pi/L$ , and the length of a wave in water of finite depth is a function of its frequency and the depth of the water. (A similar equation to get the spectrum of  $\dot{u}(t)$  can also be written down.) One can look these functions up in tables and the only variable for a particular problem is  $\omega$  when  $d$  and  $D$  are fixed. The result of multiplying the spectrum in Figure (5) by equation (10) and by a similar equation to get  $\dot{u}$  is shown in Figure (6). What happens is quite evident because the hyperbolic cosine for waves in very deep water becomes an exponential,  $e^{-kd}$ , and the high frequencies are attenuated.

At a certain frequency the spectrum is cut off very sharply. The frequency corresponds to waves with the length equal to twice the depth of the observation point.

A simple example would be a depth of 15 feet with a length of under 30 feet for the waves. From  $L = 5.12 T^2$ , the period is slightly over 2 seconds. For any frequency corresponding to  $2\pi/2$  or higher there is just nothing left in the spectrum of the  $u$  component.

Spectra for  $u(t)$  and  $\dot{u}(t)$  with different dimensions (velocity squared times time and acceleration squared times time, respectively) are the result. All of the high frequency contributions to  $\eta(t)$  have been lost in the spectra of  $u$  and  $\dot{u}$ . The area under the spectral curve for  $u(t)$  is equal to the variance of the  $u$  component of the particle velocities. It is:

$$\int_0^{\infty} S_u(\omega) d\omega = \psi_1 \quad (11)$$

This is the variance of the horizontal component of the velocity at this depth. The record of  $u(t)$  did not have to be constructed to obtain this result. An operation on a spectrum was all that was needed.

Now by taking this spectrum of  $u(t)$  and performing some operations on it, one can predict many statistical properties of  $u(t)$  such as the average time between zeros and the highest 10 percent of the  $u$  velocities. These properties can be predicted because  $u(t)$  is also a stationary Gaussian process. What is the advantage of doing it this way? The advantage is that these are the quantities one wants and it costs a lot to get them. The spectrum can usually be described by about 60 to 120 numbers instead of 2400 time points for a typical 20 minute wave record. The transformation required to get the spectrum of  $u(t)$  is the result of a multiplication of the wave spectrum by somewhere between 60 to 120 numbers. The statistical properties of this  $u$  component of the velocity can be predicted without leaving the spectral domain. One calls the transformation a "response operator" in ship motion studies. Incidentally there is no phase shift. There is no problem of time delay. In this particular case each component remains in phase with the surface component. The other way to get the same information about  $u(t)$  would be to apply equation (7) to  $\eta(t)$  to produce a 2400 point record of  $u(t)$  and then extract the required statistical information about  $u(t)$  from its time history. This would require, say, 100 to 120 multiplications in each convolution to get  $u(t)$  at each time point and there are, say 2400 points in the record so that from 240,000 to 360,000 arithmetic operations would be needed as contrasted with perhaps 60 to 120 in this case and perhaps 10 times that to apply the probability results of Rice.

The  $\dot{u}$  spectrum is handled in just the same way except

the  $\dot{u}$  is 90 degrees out of phase with  $u$ .<sup>‡</sup> One wants to predict the spectrum of  $\dot{u}$ . The filter function that has just been discussed simply has to be multiplied by  $\omega^2$ . What is the spectrum of  $\dot{u}$ ? It is also shown in Figure 6 and defined by

$$S_{\dot{u}}(\omega) = \omega^2 S_u(\omega) \quad (12)$$

The high frequencies are amplified by this filter function and the low frequencies are attenuated. Then one could compute

$\psi_2 = \int_0^\infty S_{\dot{u}}(\omega) d\omega$  as the variance of the time derivative of the velocity.

What does the spectrum of  $\dot{u}$  tell one about  $\dot{u}$ . The spectrum indicates that there ought to be more zeros and shorter "periods" in the record for  $\dot{u}$  than in the record of the velocity at this same depth. This one would find by applying certain results given in Rice (1944). The horizontal acceleration of the water at that same depth has a different structure from the horizontal velocity of the water at that same depth.

Approximately 2400 points in the original time history would be involved in computations to get these same results in the time domain, but now a stage has been reached where computations on only 60 to 120 numbers are needed to get similar results concerning the probabilistic structure of the record. These results cannot give anything about the exact time history of  $u$  or the exact time history of  $\dot{u}$ , but still one can obtain a wide variety of results about things like the variance of  $u$ , the variance of  $\dot{u}$ , the zero crossings of  $u$ , the probability structure of  $u$  -- everything that you can deduce from the spectra. This is why one works with spectra. In a sense one wants to generalize; one only wants certain features. If these features are not the ones that are wanted,

<sup>‡</sup> A little more care at this point would introduce the appropriate cross spectra.

one does not proceed this way. But, if probability results are wanted, one looks at the power spectrum of the time history that is being studied.

A lot of our work has been to do exactly that, namely, to study what can be gotten out of the spectral representation. One can never say anything specific about any particular record, but one can say a lot of things about the statistical properties of the record or any other one from the same population.

Every time one changes any one of the parameters in the problem of wave forces on a pile, one has to recompute. Whether in the time domain or in the spectral domain, this is neither an advantage or a disadvantage. In one way, spectral techniques are advantageous because the amount of re-computation required to cover a wide variety of conditions is less than in doing the analysis in the time domain.

However, by these spectral techniques, the results of Reid have still not yet been obtained. So far, just the spectra of  $u$  and  $\dot{u}$  have been obtained with the possibility of deducing many of the statistical properties of  $u$  and  $\dot{u}$ . Nothing about what the function,  $f(t)$ , looks like has yet been obtained.

With reference to the wave record,  $\eta(t)$ , every time anyone has taken a wave record at the surface of the sea in which they believed the recording procedure was accurate and studied it in order to see if it is normal, something was found that looked very close to normal. However, as soon as any tests are tried on it, the results suggest that the record is not quite normal. The non-linear features of waves at the free surface are probably the reason why perfect agreement is not reached. Theoreticians do not know how to handle these non-linear features yet, although this is one of the things that I am working on now.\* But for many, many applications, what we have is more than enough. The

\*To up date this remark see the papers given at the Eastern Conference on Waves to appear in April 1963. (National Academy Science 1963)

record is so close to normal that usually one does not make a serious error by assuming the waves are normal. For the velocities below the surface, the assumption of normality is even closer to actuality.

Now, what could a statistician say about the function  $f(t)$ ? The first thing he would say is that it is not Gaussian, because it contains a square of a Gaussian distribution in the  $|u|u$  term. The problem of the distribution of  $f(t)$  has been studied a little, and I can show you some things about it as defined by equation (1).

What is needed is the distribution of  $f$ , given the joint distribution of  $u$  and  $\dot{u}$ . What is the distribution of  $u$  and  $\dot{u}$ ? We know that  $u$  at any particular instant of time is a number from a normal distribution with a variance of  $\psi_1$ , and points read from  $\dot{u}$  are numbers from a normal distribution with a variance of  $\psi_2$ . Also from a probability point of view  $u$  and  $\dot{u}$  are independent. That is, the value of  $\dot{u}$  that one will observe at this particular depth of water at a particular time has nothing to do with the value of  $u$  at that same time. This is apparent since one involves a sine and the other a cosine in equations (3) and (4). There is no correlation between them. If one takes a whole sequence of observations of pairs of values of  $u$  and  $\dot{u}$  and plots them in a scatter diagram, there will be no correlation. Thus the joint density of  $u$  and  $\dot{u}$  is given by equation (13). If  $u$  and  $\dot{u}$  were dependent, there would be a correlation coefficient and a cross product term in the exponent of (13) which would not bother further analysis too much.

$$P(u, \dot{u}) du d\dot{u} = \frac{1}{2\pi \sqrt{\psi_1 \psi_2}} \left[ e^{-u^2/2\psi_1} \right] \left[ e^{-\dot{u}^2/2\psi_2} \right] du d\dot{u} \quad (13)$$

$$-\infty < u < \infty$$

$$-\infty < \dot{u} < \infty$$

$\eta(t)$  and  $u(t)$  are highly correlated, but  $\eta(t)$  and  $\dot{u}$ , and  $u$  and  $\dot{u}$  have zero correlations at zero time lag. (However, if one compares  $u$  with  $\dot{u}$  observed, three seconds later, there is a correlation.)

Let us pursue this a little bit just to show what kind of thinking a statistician might do in this particular case. We have this function,  $f(t)$ ; let's call it  $f$ . The distribution of  $u$  and  $\dot{u}$  is defined from minus infinity to plus infinity with variances,  $\psi_1$  and  $\psi_2$  as in equation (13).

(For example, for a normal distribution with a variance of one, the odds are about one in ten thousand that one will get a value for an observation outside of plus or minus four.) Although these ranges are infinite, all observed values of  $u$  are finite for a practical case. So now one wishes to make a transformation on equation (13) that will preserve measure, that will still be a probability density function, and that will give the distribution of  $f$ .

The random variable,  $f$ , has to be defined in two separate regions. It has to be defined for  $0 < u < \infty$  and for  $-\infty < u < 0$ . This absolute value sign in equation (1) has a different meaning depending upon the sign of  $u$  and so let us assume that  $u < 0$  and then  $|u| = u$  so that

$$f = k_1 u^2 + k_2 \dot{u} \quad (14)$$

In the plane of  $u$  and  $\dot{u}$ , one is interested in the right half plane as in Figure (7).

To make a transformation of variables, define

$$\dot{u}^* = \dot{u} \quad (15)$$

(this is a simple little device to keep the transformation of variables straight). The inverse of equations (14) and (15) is given by equations (16) and (17)

$$\dot{u} = \dot{u}^* \quad (16)$$

$$u = + \sqrt{\frac{f}{k_1} - \frac{k_2}{k_1} \dot{u}^*} \quad (17)$$

subject to the condition that  $f < k_2 \dot{u}^*$ . The Jacobian of  $\dot{u}$  and  $u$  is given by

$$\frac{\partial(u, \dot{u})}{\partial(f, \dot{u}^*)} = \frac{1}{2k_1} \left[ \frac{1}{\sqrt{\frac{f}{k_1} - \frac{k_2}{k_1} \dot{u}^*}} \right] \quad (18)$$

and with these results equation (13) can be transformed into equation (19)

$$P_1(f)df = \frac{1}{2\pi \sqrt{\psi_1 \psi_2}} \left[ \int_{-\infty}^{f/k_2} \frac{f/k_2 \left( -\frac{1}{2\psi_1} \left[ \frac{f}{k_1} - \frac{k_2}{k_1} \dot{u}^* \right] - \frac{1}{2\psi_2} (\dot{u}^*)^2 \right)}{2k_1 \sqrt{\frac{f}{k_1} - \frac{k_2}{k_1} \dot{u}^*}} d\dot{u}^* \right] df \quad (19)$$

An additional transformation of variable, namely that

$$\alpha = \frac{f}{k_1} - \frac{k_2}{k_1} \dot{u}^* = u^2 \quad (20)$$

So that

$$d\alpha = -\frac{k_2}{k_1} d\dot{u}^* \quad (21)$$



yields

$$P_1(f)df = \left[ \frac{1}{2\pi \sqrt{\psi_1 \psi_2}} \int_0^\infty \left( \frac{e^{-\frac{\alpha}{2\psi_1} - \frac{1}{2\psi_2} \left( \frac{f}{k_2} - \frac{k_1 \alpha}{k_2} \right)^2}}{2k_2 \sqrt{\alpha}} \right) d\alpha \right] df \quad (22)$$

The other half of the distribution is obtained by considering  $-\infty < u < 0$ . For this condition,

$$f = -k_1 u^2 + k_2 \dot{u} \quad (23)$$

$$\dot{u}^* = \dot{u} \quad (24)$$

A similar sequence of operations yields the result that

$$P_2(f)df = \left[ \frac{1}{2\pi \sqrt{\psi_1 \psi_2}} \int_0^\infty \left( \frac{e^{-\frac{\alpha}{2\psi_1} - \frac{1}{2\psi_2} \left( \frac{k_1 \alpha}{k_2} + \frac{f}{k_2} \right)^2}}{2k_2 \sqrt{\alpha}} \right) d\alpha \right] df \quad (25)$$

The sum of equation (22) and (25) is given by

$$P(f)df = \frac{1}{2\pi \sqrt{\psi_1 \psi_2}} \left[ \int_0^\infty \left( \frac{e^{-\frac{\alpha}{2\psi_1} - \frac{1}{2\psi_2} \left( \frac{k_1 \alpha}{k_2} + \frac{f}{k_2} \right)^2}}{2k_2 \sqrt{\alpha}} \right) d\alpha \right]$$

$$+ \int_0^\infty \frac{e^{-\frac{\alpha}{2\psi_1}} - e^{-\frac{1}{2\psi_2} \left( \frac{k_1}{k_2} - \frac{f}{k_2} \right)^2}}{2k_2 \sqrt{\alpha}} d\alpha \quad df \quad (26)$$

As a check, equation (26) can be integrated over  $f$  from  $-\infty$  to  $\infty$  prior to the integration over  $\alpha$ . The integration over  $\alpha$  is then simple and one obtains the result finally that

$$\int_{-\infty}^{\infty} P(f) df = 1 \quad (27)$$

(If  $k_1$  is a function of  $u$ , then  $f$  is no longer just a quadratic function of  $u$ , it's a function of  $u$  that is even more complicated. On the assumption that  $k_1$  is some constant for the range of  $u$  that has occurred then the above derivation is useful.) The book to read on such transformations is Courant - Differential and Integral Calculus. What does equation (26) tell us? If you knew  $k_1$ ,  $k_2$ ,  $\psi_1$ , and  $\psi_2$ , you could evaluate this integral as a function of  $\alpha$  to get the function of  $f$  out of it. Then you could plot the probability density as  $f$  went from  $-\infty$  to  $+\infty$ . If one cannot evaluate the integral, one writes a program for it on a computing machine.  $P(f)$  can be computed in a few minutes. Anyway one can get  $P(f)df$ , one way or the other, from equation (26).

For one condition, if  $k_2 \hat{u}$  were very large compared to  $k_1 u |u|$ ,  $P(f)df$  would be nearly a normal distribution; if  $k_2 \hat{u}$  were unimportant compared to  $k_1 u |u|$ ,  $P(f)df$  would be one half of a  $\chi^2$  (chi square) distribution with one degree of freedom reflected in the origin. For intermediate cases, depending upon the relative strength of  $k_1 u |u|$  and  $k_2 \hat{u}$ , one gets a whole family of curves. Just what  $P(f)df$  would look like I am not sure, but whatever it looks like, it

would not be normal if  $k_1 u |u|$  makes any contribution at all. (The  $\chi^2$  distribution and the Gamma distribution are simply two names for the same thing; see Mood (1950) Introduction to the Theory of Statistics, McGraw-Hill Book Company, Inc.)

What one could do next would be to go through a recording of  $f(t)$ , if it has been recorded, read off the values of that function at equally spaced times, plot up the histogram, and compare it with this distribution (equation (26)) to see if the theory checks out.

We have now reached the point where certain properties of  $f(t)$  have been obtained without ever evaluating records of  $u$  and  $\dot{u}$ . One can start with a free surface record as a function of time, compute the spectrum of it, operate on the spectrum to compute the spectra of  $u$  and  $\dot{u}$ , take the areas under those two spectra to get the variances of  $u$  and  $\dot{u}$ , and then from equation (26) predict the distribution of individual observations of  $f(t)$ .

Every time the spectrum of the free surface is changed one gets a new distribution. Now if the distribution of  $f(t)$  is what you want to get, it is a lot quicker this way, and in many ways not quite so hard, than to do the time domain study starting with the wave record.

In the above derivation it has been assumed that  $k_1$  and  $k_2$  are constants that are known. If everything else in equation (26) were known except  $k_1$  and  $k_2$ , actual observations of  $P(f)$  and computations of  $\psi_1$  and  $\psi_2$  could possibly then be used to find out what the values of  $k_1$  and  $k_2$  ought to be to give the best fit to the observed  $P(f)$ .

These results give you the probability structure of  $f(t)$  at one depth. If one wants to go to another depth, one has to do it over again. Each different spectral component is attenuated at a different rate at the new depth. But once one has a computer program, something like changing parameters is nothing, one just recomputes with the new

parameters. Once one goes through the trouble of writing a proper computer program one can do any of these things in either the time domain or in the spectral domain. Once you have a program, the processing of a large amount of data is easy. A little bit of patience and these computers zip right through this material with no trouble at all. One can vary the parameters all over the place to get some sort of a feeling for design considerations.

The derivation of the distribution of  $P(f)$  would be one part of the answer in studying  $f(t)$ . But one wants to know other things too, and these will be very hard to derive.  $f(t)$  is a stationary process, but certainly not Gaussian. What does the spectrum of  $f(t)$  mean? It only represents a part of what is needed to describe  $f(t)$ .

If this theoretical representation is inadequate for  $f(t)$ , then the engineer must produce one that is adequate before these techniques can be applied. In a sense that is an engineer's problem not mine. But if equation (1) is an adequate expression for  $f(t)$ , then this is the way the problem is handled if one is interested in it from a probability point of view.

But if  $f(t)$  is not adequately represented by equation (1), then we approach the game of musical chairs. One runs into this a lot today. One solves a problem that needs to be solved and goes to the people one is working with and they wonder at your answer. Then they suggest changing the problem. One obviously has got to start with a properly formulated problem that is adequate from a physical point of view. If one formulates a nonsense problem, one is guaranteed to get nonsense results. If one does not formulate a problem that corresponds to physical reality, all the mathematics in the world is not going to tell you anything sensible about you final result.

I like to solve a problem that someone thinks has a reasonable chance of being applicable to a practical problem in waves or ship motions, or what have you. These problems are the hard ones to solve because one can go around in circles for a long, long time with different people trying to set down the rules with which one is going to operate.

One always seems to run into someone that says that such and such a term has been neglected and that the problem cannot possibly correspond to reality. If one cannot put reality into the problem at the start, one surely does not get it out. If one cannot put the pertinent physical parameters, the structure of the problem, in mathematical terms, all the operations on it in the world aren't going to get out anything sensible.

But look at the beauty of just the derivations of going from the free surface to the horizontal velocity components here. There is physical reality in this step. It's the physical attenuation of each spectral component according to frequency for irregular records. This is the type of theory that the classical hydrodynamicist derived before 1920; they had all of this well understood for sine waves, but the trouble was that ocean waves are not sine waves. If ocean waves were sinusoidal, there would not be any problem, but they are not.

#### CONCLUDING APPENDED REMARKS

Since the above lecture was given, Mr. Seymour Kaplan of New York University has studied equation (26) in greater detail and shown how it could be evaluated as a probability density function. His results are given in Appendix I.

There are a number of possible applications of these results the most direct being the study of  $f(t)$  as observed in a wave tank in which long crested irregular random waves are generated as at Davidson Laboratory. This would provide a useful check of the consistency of the values for the constants in equation (1) and a check of the correctness of the above probability density function.

It might be possible to extend the analysis to short crested waves by considering  $u$ ,  $v$ ,  $\dot{u}$  and  $\dot{v}$ .

Finally what is ultimately needed is a quantity of the form

$$F(t) = \int_{-l}^0 (l' - z)f(t)dz \quad (28)$$

to get the bending moment on the pile, for example. This could be put in the form

$$F(t) = \sum_{m=0}^n (l' + l - m\Delta l) f_m(t) \quad (29)$$

where  $f_m(t) = f(t : z = -l + m\Delta l)$  ( $z$  is a parameter).

The different  $f_m(t)$  would have a complicated multivariate density structure, but it would perhaps be possible to obtain some results on the distribution of the actual integrated effect on the pile.

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APPENDIX I  
EVALUATION OF THE PROBABILITY DENSITY OF THE FORCE  
by  
Seymour Kaplan



The probability density is of the form:

$$P(f) = \frac{1}{2\pi \sqrt{\psi_1 \psi_2}} \left\{ \int_0^\infty \frac{e^{-\alpha/2\psi_1} \left[ e^{-\left[\frac{k_1\alpha}{k_2} + \frac{f}{k_2}\right]^2/2\psi_2} \right]}{2k_2 \sqrt{\alpha}} d\alpha \right. \\ \left. + \int_0^\infty \frac{e^{-\alpha/2\psi_1} \left[ e^{-\left[\frac{k_1\alpha}{k_2} - \frac{f}{k_2}\right]^2/2\psi_2} \right]}{2k_2 \sqrt{\alpha}} d\alpha \right\}$$

$$-\infty < f < \infty$$

This breaks up into two integrals:

$$I_1 = A e^{-\gamma_3(f)} \int_0^\infty \frac{e^{-\gamma_1 \alpha^2 - \gamma_2 \alpha}}{\sqrt{\alpha}} d\alpha$$

$$I_2 = A e^{-\gamma_3'(f)} \int_0^\infty \frac{e^{-\gamma_1' \alpha^2 - \gamma_2' \alpha}}{\sqrt{\alpha}} d\alpha$$

where

$$A = 1/4 \pi k_2 \sqrt{\psi_1 \psi_2}$$

$$\gamma_1 = k_1^2/2\psi_2 k_2^2$$

$$\gamma_2(f) = \left[ \frac{fk_1}{\psi_2 k_2^2} + \frac{1}{2\psi_1} \right]$$

$$\gamma_3(f) = f^2/2\psi_2 k_2^2$$

$$\gamma_1' = k_1^2/2\psi_2 k_2^2$$

$$\gamma_2'(f) = \left[ \frac{-fk_1}{\psi_2 k_2^2} + \frac{1}{2\psi_1} \right]$$

$$\gamma_3'(f) = f^2/2\psi_2 k_2^2 = \gamma_3(f)$$

$I_1(f)$  and  $I_2(f)$  are evaluated easily if we note that the integrands of  $I_1$  and  $I_2$  are the Laplace transforms of

$$\frac{e^{-\gamma_1 \alpha^2}}{\sqrt{\alpha}} \quad \text{and} \quad \frac{e^{-\gamma_1' \alpha^2}}{\sqrt{\alpha}} .$$

Referring to an extensive table of Laplace transforms,\* we find that

\* Tables of Integral Transforms, Volume I, Bateman Manuscript Project, page 146, equation (23).

$$I_1 = \frac{A}{2} \sqrt{\frac{\gamma_2}{\gamma_1}} e^{-\gamma_3} + \frac{\gamma_2^2}{8\gamma_1} K_{\frac{1}{4}}\left(\frac{\gamma_2}{8\gamma_1}\right)$$

and similarly,

$$I_2 = \frac{A}{2} \sqrt{\frac{\gamma_2'}{\gamma_1'}} e^{-\gamma_3} + \frac{(\gamma_2')^2}{8\gamma_1'} K_{\frac{1}{4}}\left(\frac{(\gamma_2')^2}{8\gamma_1'}\right)$$

The functions  $K_\nu(x)$  are called Bessel functions of imaginary argument, are extensively tabulated, and are fundamental solutions of the differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2) y = 0.$$

The particular function desired is  $K_{\frac{1}{4}}(x)$  where  $x$  is real.  $K_{\frac{1}{4}}(x)$  is always positive, tends exponentially to zero as  $x \rightarrow \pm \infty$ , and therefore has no zeros for  $-\infty < x < \infty$ . It is an even function of  $\nu$ , i.e.,  $K_{-\nu}(x) = K_\nu(x)$ .

An asymptotic expansion for  $K_\nu(x)$  is:

$$K_\nu(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left[ 1 + \frac{(4\nu^2 - 1^2)}{1!8x} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2!(8x)^2} + \dots \right]$$

Values of  $K_{\frac{1}{4}}(x)$  may be found from tables of  $I_{\frac{1}{4}}(x)$  and  $I_{-\frac{1}{4}}(x)$  and the relation:\*

$$K_{\nu}(x) = \frac{\pi}{2\sin\pi\nu} [I_{-\nu}(x) - I_{\nu}(x)]$$

Presumably, one would want to calculate  $P(f)$  as a function of  $f$  for various sets of the parameters  $\psi_1$ ,  $\psi_2$ ,  $k_1$ , and  $k_2$ .

Thus, a set of parameter values would be the input to the problem and the plot of  $P(f)$  vs  $f$  would be the output.

For  $f \rightarrow 0$ ,  $\gamma_2 \rightarrow \gamma_2' = 1/2\psi_1$  and  $\gamma_3 \rightarrow \gamma_3' = 0$ . Therefore:

$$P(0) = \frac{A}{2} \sqrt{\frac{\psi_2 k_2^2}{k_1^2 \psi_1}} e^{\psi_2 k_2^2 / 16 \psi_1^2 k_1^2} K_{\frac{1}{4}}\left(\frac{\psi_2 k_2^2}{16 \psi_1^2 k_1^2}\right) + \frac{A}{2} \sqrt{\frac{\psi_2 k_2^2}{k_1^2 \psi_1}} e^{\psi_2 k_2^2 / 16 \psi_1^2 k_1^2} K_{\frac{1}{4}}\left(\frac{\psi_2 k_2^2}{16 \psi_1^2 k_1^2}\right)$$

As  $f$  gets large,  $P(f)$  should go to zero. It is helpful to see what  $P(f)$  looks like for large  $f$ . The  $\sqrt{\frac{\gamma_2}{\gamma_2}}$  becomes  $\sqrt{\frac{2f}{k_1}}$ . As can be seen from the asymptotic expansion,

\* Tables of Bessel Functions of Fractional Order, Volume II, Computation Laboratory, National Bureau of Standards, Columbia Univ. Press, 1949.

$$K_{\frac{1}{4}}(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \quad \text{and}$$

$$I_1(f) \sim \frac{A}{2} \sqrt{\frac{2f}{k_1}} e^{-\gamma_3 \left(\frac{\pi 8 \gamma_1}{2 \gamma_2}\right)^{\frac{1}{2}}} = A \left(\frac{\pi \psi_2 k_2^2}{f k_1}\right)^{\frac{1}{2}} e^{-f^2/2 \psi_2 k_2^2}$$

A similar formula holds for  $I_2(f)$ . The above approximation would probably be good for numerical values of  $f \geq 5$ .

For smaller values of  $f$ , the tables given in the second footnote could be read into the computer and a program involving a table look-up can be easily written for the evaluation.

Also, since  $P(f)$  is a probability density, it cannot be negative. This is obvious from the equations for  $I_1$  and  $I_2$ , remembering that  $K_{\frac{1}{4}}(x)$  has no real zeros. After an initial

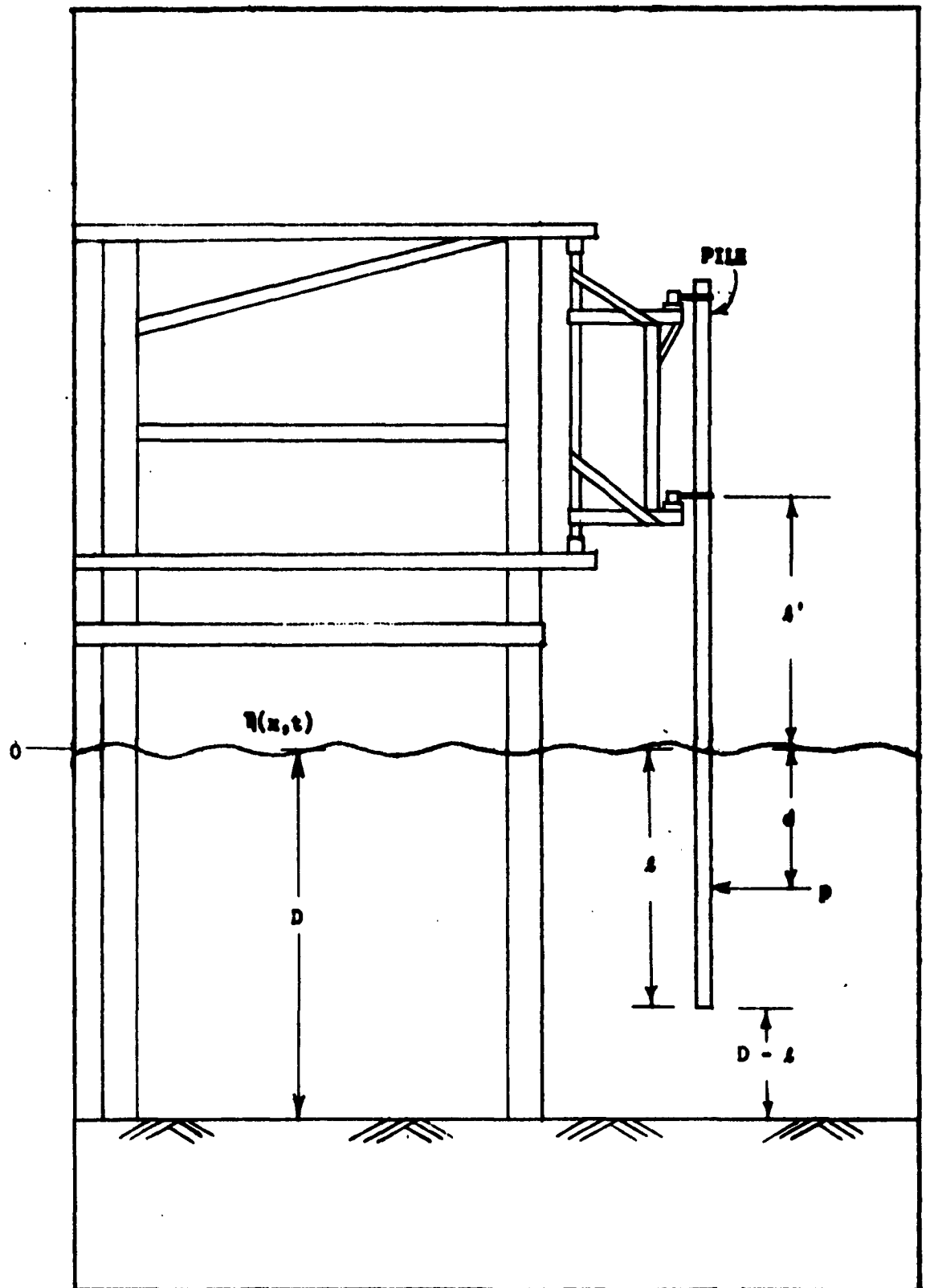
set of runs, a check should be made to insure that  $\int_{-\infty}^{\infty} P(f) df = 1$ , as is necessary if  $P(f)$  is to be a probability density.

APPENDIX II

ATTENUATION FUNCTION FOR VELOCITY  $u$ ,  
AND ACCELERATION  $\dot{u}$ .

Water Depth (D)		Depth of Submergence (d)		
97.50		74.50		
$L_o$	Frequency, $\omega$ (rps)	Period, T (sec.)	$f(u)$	$f(\dot{u})$
10.0	4.4951	1.397	.0000	.0000
20.0	3.1784	1.976	.0000	.0000
30.0	2.5952	2.421	.0000	.0000
40.0	2.2475	2.795	.0000	.0000
50.0	2.0102	3.125	.0000	.0000
60.0	1.8350	3.423	.0000	.0000
70.0	1.6990	3.698	.0000	.0000
80.0	1.5892	3.953	.0000	.0000
90.0	1.4984	4.193	.0000	.0001
100.0	1.4214	4.420	.0001	.0003
150.0	1.1606	5.413	.0033	.0045
200.0	1.0051	6.250	.0138	.0140
250.0	.8990	6.988	.0311	.0252
300.0	.8206	7.656	.0518	.0340
350.0	.7598	8.268	.0728	.0420
400.0	.7107	8.840	.0925	.0467
500.0	.6357	9.883	.1262	.0510
600.0	.5803	10.827	.1528	.0514
700.0	.5372	11.694	.1736	.0501
800.0	.5025	12.501	.1903	.0480
900.0	.4738	13.260	.2038	.0457
1000.0	.4495	13.977	.2150	.0434
1200.0	.4103	15.312	.2324	.0391
1400.0	.3799	16.538	.2452	.0353
1600.0	.3553	17.680	.2550	.0322
1800.0	.3350	18.752	.2627	.0294
2000.0	.3178	19.766	.2690	.0271
2500.0	.2842	22.100	.2804	.0226

3000.0	.2595	24.210	.2882	.0194
4000.0	.2247	27.953	.2980	.0150
5000.0	.2010	31.256	.3039	.0122
6000.0	.1835	34.237	.3079	.0103
7000.0	.1699	36.980	.3108	.0089
8000.0	.1589	39.534	.3129	.0079
9000.0	.1498	41.930	.3146	.0070
10000.0	.1421	44.201	.3160	.0063
20000.0	.1005	62.509	.3221	.0032



**FIGURE 1.** Configuration of pile, bottom and wave surface with notation.



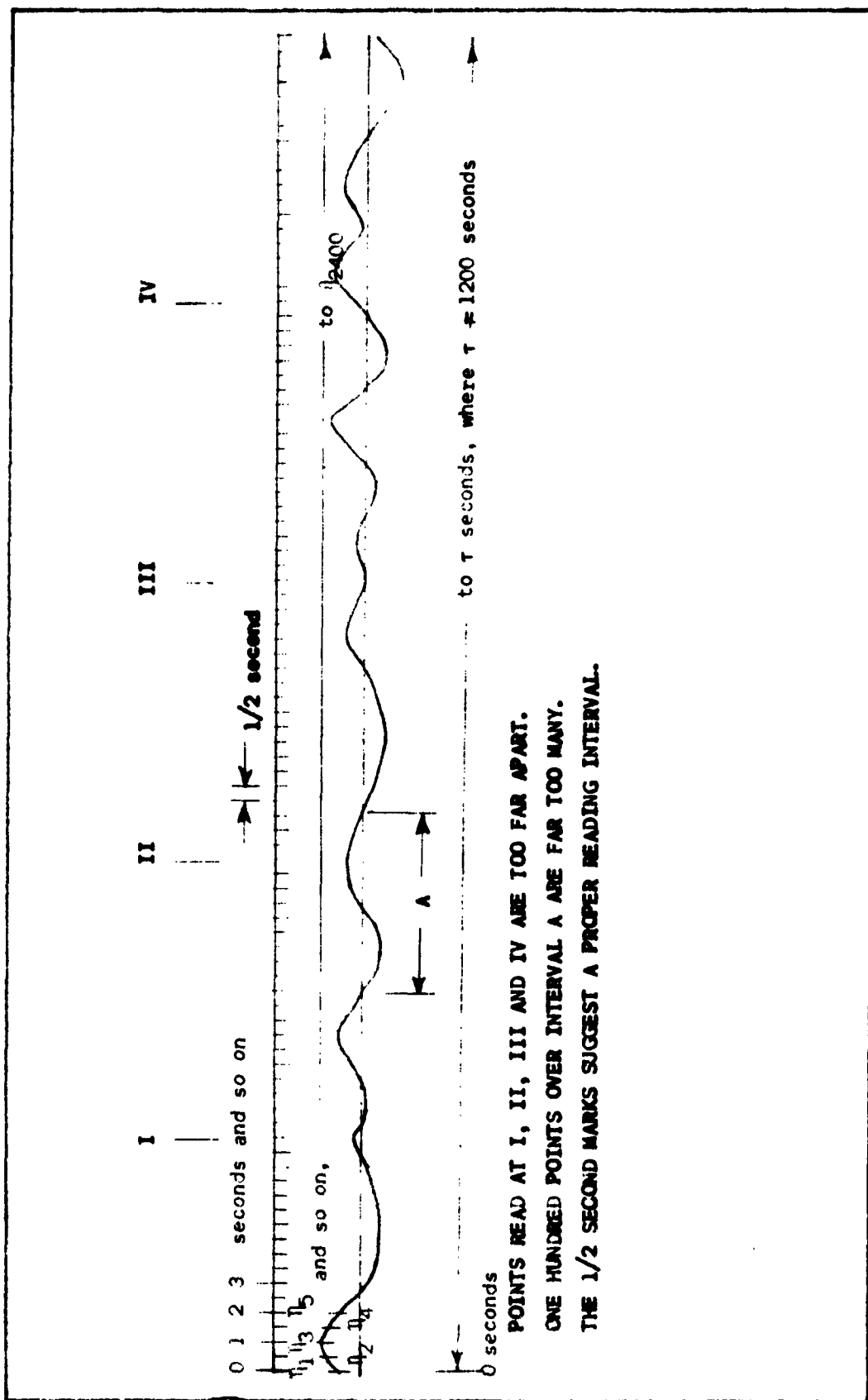


FIGURE 2. Free surface as a function of time to show digitization interval.

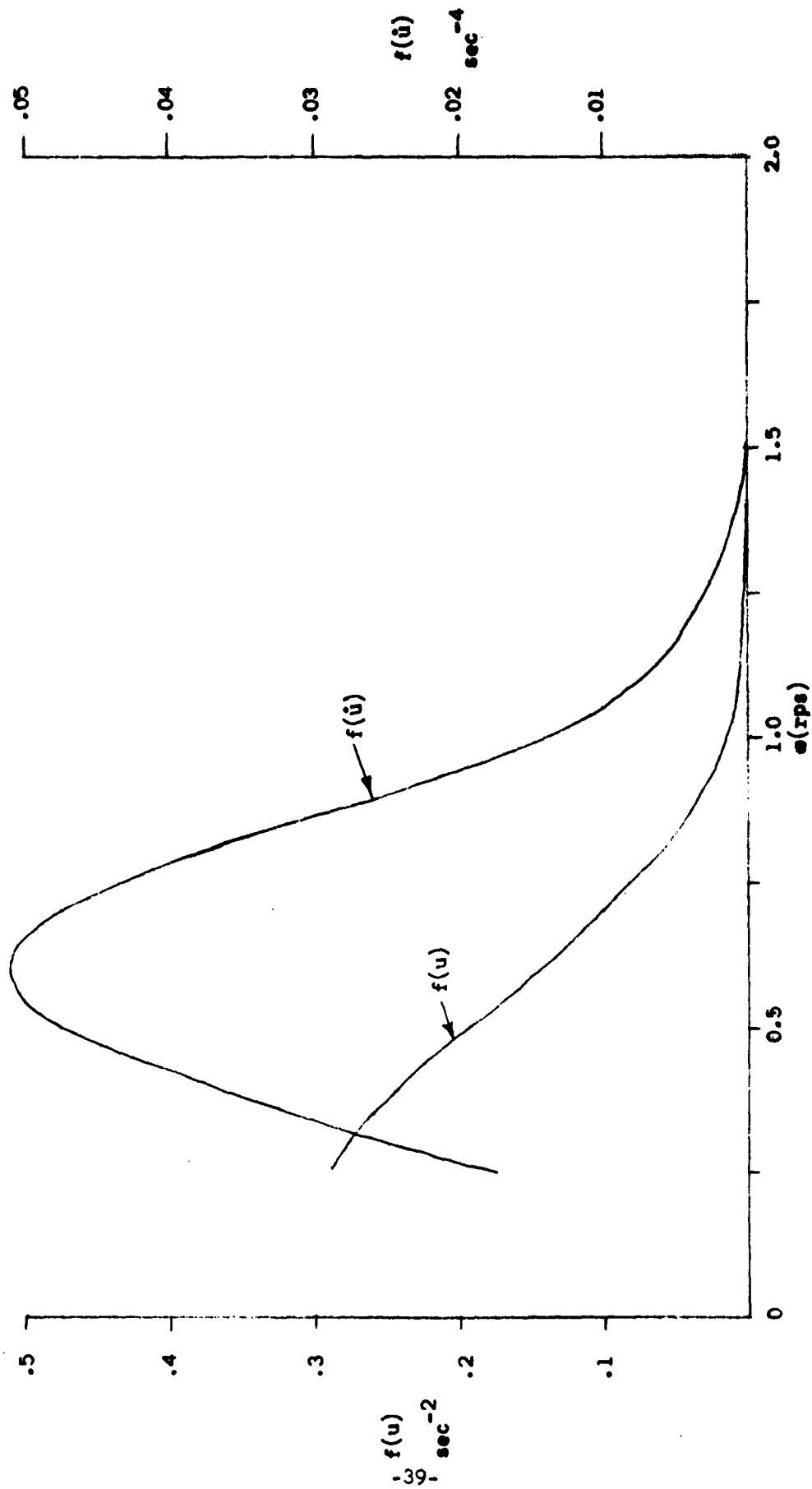


Figure 3. Attenuation Functions for Velocity  $u$  and acceleration  $\dot{u}$  for  $D = 97.7$  and  $Z = 74.51$ .

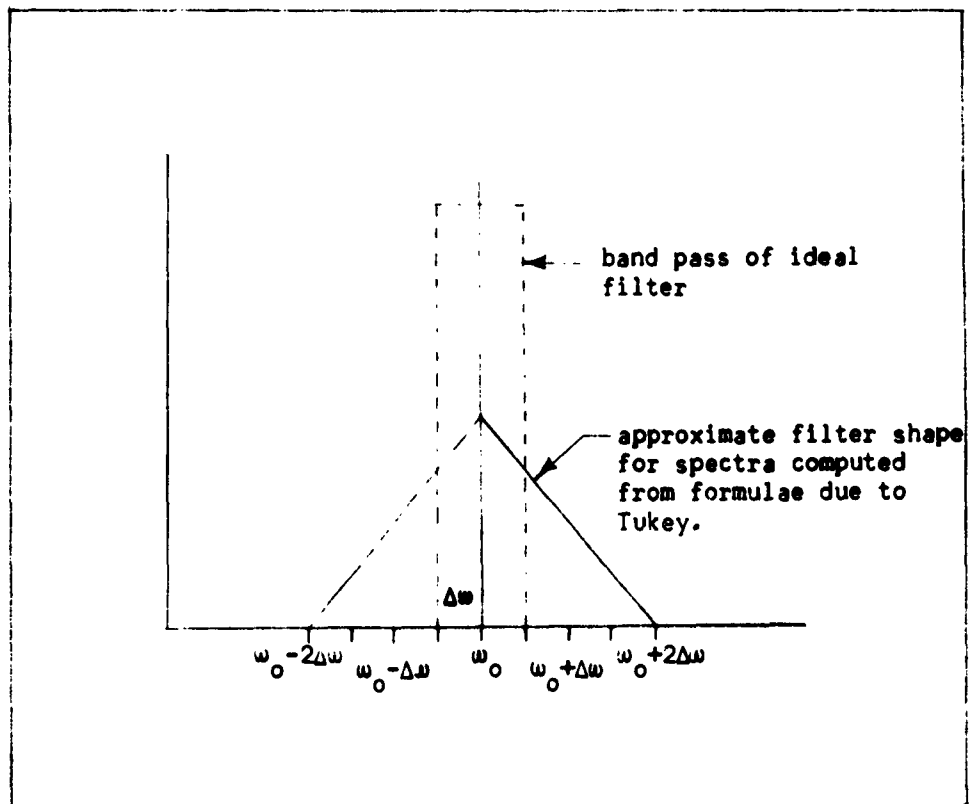


Figure 4. Example of narrow band filter.

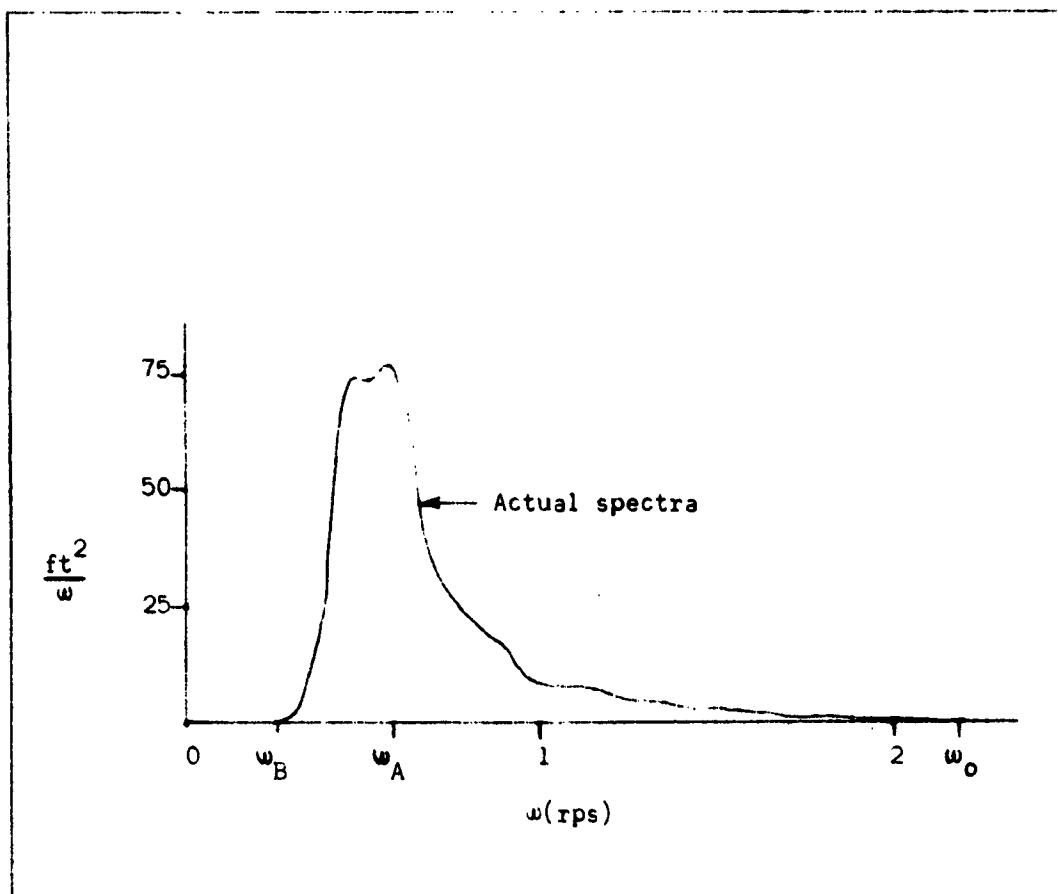


Figure 5. Sketch of power spectrum (free surface) as a function of frequency.

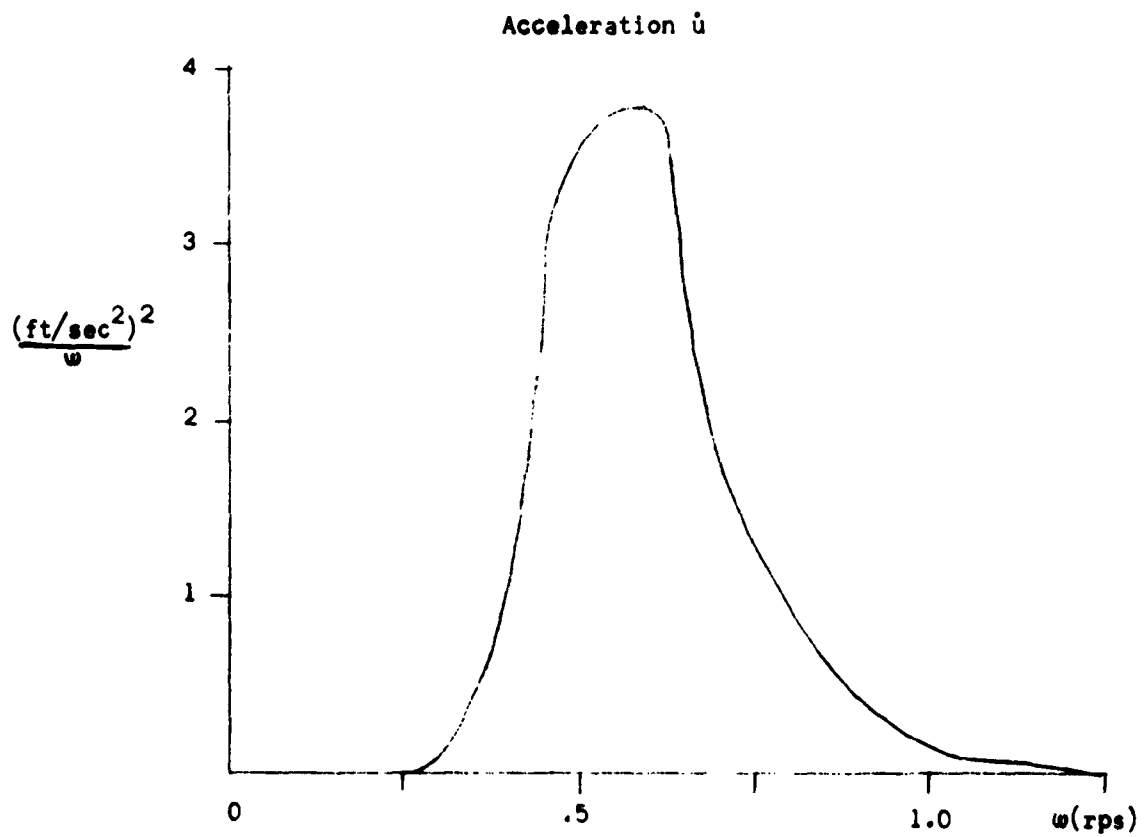
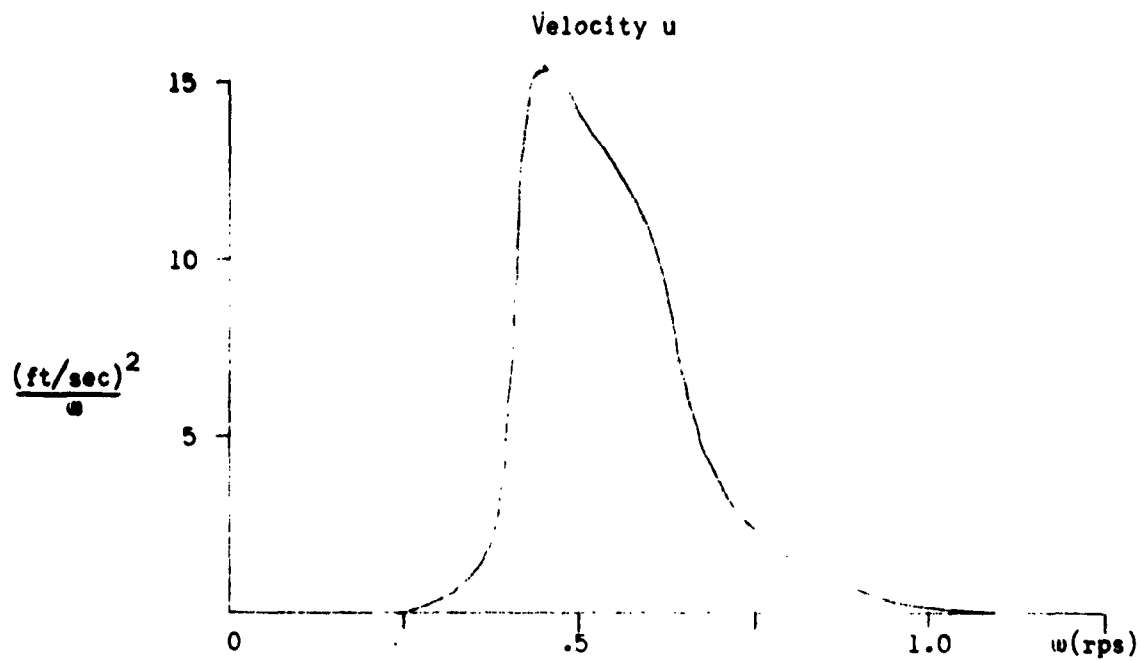


Figure 6. Spectra of velocity  $u$ , and acceleration  $\dot{u}$ .

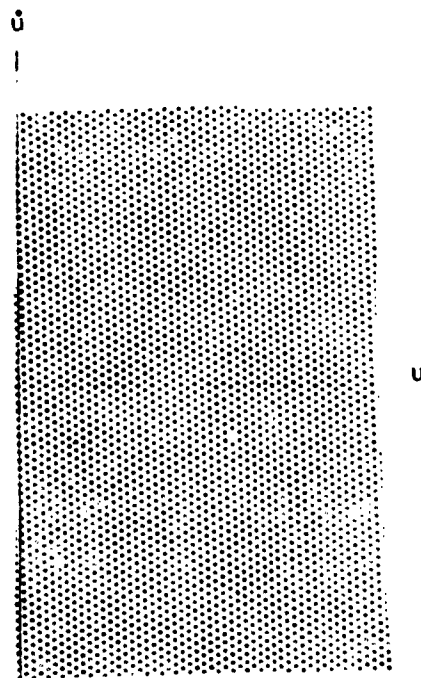


Figure 7 Sketch of  $u - \dot{u}$  plane